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CONVERGENCE OF SERIES IN SOME MULTI-INDEX MITTAG-LEFFLER FUNCTIONS ⋆

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Abstract: In this paper Cauchy-Hadamard, Abel, Tauber and Littlewood type theorems for series in some multi-index Mittag-Leffler functions are given. We provided also asymptotic formulae for "large" values of indices of these functions, used in the proofs of the convergence theorems for the considered series.

Keywords: multi-index Mittag-Leffler functions, Cauchy-Hadamard, Abel, Tauber and Littlewood type theorems, summation of divergent series, asymptotic formula

1. INTRODUCTION

The Mittag-Leffler (M-L) functions E_α and $E_{\alpha,\beta}$ are entire functions of $z \in \mathbb{C}$ defined by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{1.1}$$

$$\alpha > 0,\ \beta > 0,$$

and are natural extensions of the exponential function and trigonometric functions like cos-function. A description of their basic properties appeared back in the Bateman Project [6], "Higher Transcendental Functions" (Vol. 3), in a chapter devoted to "miscellaneous functions". The functions (1.1) have been studied in details by Dzrbashjan [5]. The detailed properties of these functions can be found in the contemporary monographs of Kilbas et al. [8] and Podlubny [19]. The M-L functions (1.1) and their multi-index analogues (1.2) have been used as generating functions of the so-called Gel'fond-Leont'ev operators of generalized integration and differentiation (being operators of fractional calculus) by Dimovski and Kiryakova [3] and Kiryakova [9] (see details in Ch.2 and Ch.5), also in Kiryakova [10], [11].

Recently, in [23], [10],[11], [13], it has been introduced and studied (see e.g. [10], [11]) a class of special functions of Mittag-Lefler type that are multi-index analogues of $E_{\alpha,\beta}(z)$. The indices $\alpha := 1/\rho$, $\beta := \mu$ are replaced by two sets of multi-indices $\alpha \to (1/\rho_1, 1/\rho_2, ..., 1/\rho_m)$, and $\beta \to (\mu_1, \mu_2, ..., \mu_m)$.

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Definition 1.1. Let $m > 1$ be an integer, $\rho_1, ..., \rho_m > 0$ $\mu_1, ..., \mu_m$ be arbitrary real (complex) numbers. By means of these "multi-indices", the multi-index Mittag- Leffler functions (mult-M-L f-s) are defined as ([10],[11]):

$$E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\mu_1 + \frac{k}{\rho_1}\right) \ldots \Gamma\left(\mu_m + \frac{k}{\rho_m}\right)}. \tag{1.2}$$

The same functions, considered also by Luchko [13] and Yakubovich and Luchko [23], are called Mittag-Leffler functions of vector index. As proved in [21], the multi-index Mittag- Leffler functions (2.1) are entire functions of order ρ with $\frac{1}{\rho} = \frac{1}{\rho_1} + ... + \frac{1}{\rho_m}$ and type $\sigma = \left(\frac{\rho_1}{\rho}\right)^{\frac{\rho}{\rho_1}} ... \left(\frac{\rho_m}{\rho}\right)^{\frac{\rho}{\rho_m}}$.

In our previous papers [15]-[17] we studied series in systems of some representatives of special functions of fractional calculus (SF of FC) which are fractional indices analogues of the Bessel functions and also multi-index Mittag-Leffler functions (in the sense of Kiryakova [12]). There are proved Cauchy-Hadamard, Abel, Tauber and Littlewood type theorems in a complex domain.

In this paper we present some asymptotic formulae for "large" values of indices of the M-L functions (1.1) and study the convergence of series in such functions. We provide ideas of the proofs of some of the theorems and the other proofs follow the lines of the proofs in our previous papers, for series in Bessel type functions.

2. EXAMPLES OF MULTI-INDEX MITTAG-LEFFLER FUNCTIONS

i. For $m = 1$, this is the classical M-L function $E_{1/\rho,\mu}(z)$ with all its special cases, including also the exponential function $\exp(z) = E_{1,1}(z)$, cos-function $\cos z = E_{2,1}(-z^2)$, $E_{0,1}(z) = \frac{1}{1-z}$.

ii. For $m \geq 2$ is obtained the generalized Lommel-Wright function with 4 indices ($\mu > 0, q \in \mathbb{N}, \nu, \lambda \in \mathbb{C}$), introduced by de Oteiza, Kalla and Conde [14]:

$$J^{\mu,q}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^q \, \Gamma(\nu+k\mu+\lambda+1)} \tag{2.1}$$

$$= (z/2)^{\nu+2\lambda} E^{(q+1)}_{(\mu,1,\dots,1),\,(\nu+\lambda+1,\,\lambda+1,\dots,\,\lambda+1)} \left(-(z/2)^2\right).$$

This is an interesting example of a multi-index M-L function with arbitrary $m = q+1$.

iii. For $m = 2$ the functions (1.2) are Dzrbashjan's M-L type functions from [4], denoted as $E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}$. Some interesting cases are given below.

Obviously for $\underline{q = 1}$, the special function (2.1) turns into the generalization of the Bessel function $J_\nu(z)$, introduced by Pathak [18] (for details see [11], p.353 and [12], eq.(8.2)):

$$J^{\mu}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(\nu+k\mu+\lambda+1)} \tag{2.2}$$

$$= (z/2)^{\nu+2\lambda} E^{(2)}_{(\mu,1),\,(\nu+\lambda+1,\,\lambda+1)} \left(-(z/2)^2\right).$$

For particular choices of the other parameters λ and μ we obtain results for more special cases.

Let $\underline{\lambda = 0}$, then the special function (2.1) gives the generalization of the Bessel-Clifford function $C_\nu(z) = z^{-\nu/2} J_\nu(2\sqrt{z})$, introduced by E.M. Wright [22], and called Bessel-Wright or misnamed as Bessel-Maitland function (on the name E. Maitland Wright; see [12], [22]):

$$J^{\mu}_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu+\mu k+1)} = E^{(2)}_{(\mu,1),(\nu+1,1)}(-z). \tag{2.3}$$

Initially, Wright defined (2.3) only for $\mu > 0$, then extended its definition to $\mu > -1$.

Additionally if $\underline{\mu = 1}$, then (2.1) reduces to the classical Bessel function

$$J_\nu(z) = (z/2)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\Gamma(k+\nu+1)}$$

$$= (z/2)^{\nu} \, E^{(2)}_{(1,1),\,(\nu+1,\,1)}\left(-(z/2)^2\right). \tag{2.4}$$

Series in systems of functions (2.1)-(2.4) in complex domain are studied recently in [15]-[17] (Cauchy-Hadamard, Abel and Tauberian type theorems).

3. SERIES IN MITTAG-LEFFLER TYPE FUNCTIONS

We introduce auxiliary functions, associated with the Mittag-Leffler functions, adding $\widetilde{E}_{0,\beta}(z))$ and $\widetilde{E}_{\alpha,0}(z)$ just for completeness, namely:

$$\widetilde{E}_{0,\beta}(z) = 1; \quad \widetilde{E}_{n,\beta}(z) = \Gamma(\beta) z^n E_{n,\beta}(z), \tag{3.1.β}$$

$$\widetilde{E}_{\alpha,0}(z) = 1; \quad \widetilde{E}_{\alpha,n}(z) = \Gamma(n) z^n E_{\alpha,n}(z), \tag{3.1.α}$$

$$(n \in \mathbb{N}; \quad \beta > 0, \quad \alpha > 0)$$

and consider series in these functions in the complex plane, respectively of the forms:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\beta}(z), \tag{3.2.β}$$

$$\sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,n}(z), \tag{3.2.α}$$

$$\sum_{n=0}^{\infty} a_n J^{\mu,m}_{n-2\lambda,\,\lambda}(z), \quad m \in \mathbb{N}, \quad \mu > 0, \quad \lambda \in \mathbb{C} \tag{3.3}$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$.

Our main objective is to study the convergence of the series (3.2.β), (3.2.α) and (3.3) in the complex plane. Here we propose theorems, corresponding to the classical Cauchy-Hadamard, Abel, Tauber and Littlewood theorems for power series. Such kind of results are provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler functions. Convergence theorems have been obtained also for series in other special functions, for example, for series in Laguerre and Hermite polynomials [1], [2], [21], and resp. by the author – for series in Bessel functions and their Wright's 2, 3, and 4-index generalizations, see the previous papers [15] - [17].

4. CAUCHY-HADAMARD TYPE THEOREMS

First we give a theorem of Cauchy-Hadamard type for each of the above series.

Theorem 4.1. (of Cauchy-Hadamard type). *The domain of convergence of each one of the series* (3.2.β), (3.2.α), *(3.3) with complex coefficients* a_n *is the disk* $|z| < R$ *with the radius of convergence* $R = 1/\Lambda$, *where*

$$\Lambda = \limsup_{n\to\infty} \left(\, |a_n| \, \right)^{1/n}. \tag{4.1}$$

for the series (3.2.β), (3.2.α) *and there exists* $p \in \mathbb{N}_0$ *such that*

$$\Lambda = \frac{1}{2} \limsup_{n\to\infty} \left(\frac{|a_n|}{|\Gamma(\lambda+p+1)|^m |\Gamma(n-\lambda+\mu p+1)|} \right)^{1/n} \tag{4.2}$$

for the series (3.3) in Lommel-Wright functions.

The cases $\Lambda = 0$ *and* $\Lambda = \infty$ *can be included in the general case too, provided* $1/\Lambda$ *means* ∞, *respectively 0.*

5. ABEL TYPE THEOREMS

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with vertex at the point $z = z_0$, which is symmetric in the straight line defined by the points 0 and z_0.

Theorem 5.1. (of Abel type). *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers,* Λ *be the real number defined by Theorem 4.1,* $0 < \Lambda < \infty$. *Let* $K = \{z : z \in \mathbb{C}, |z| < R, R = 1/\Lambda\}$. *If* $g(z;\beta)$, $h(z;\alpha)$, $j(z)$ *are, respectively, the sums of the series* (3.2.β), (3.2.α), *(3.3) on the domain* K, *and these series converge at the point* z_0 *of the boundary of* K, *then:*

$$\lim_{z\to z_0} g(z;\,\beta) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\,\beta}(z_0), \tag{5.1.β}$$

$$\lim_{z\to z_0} h(z;\,\alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\,n}(z_0). \tag{5.1.α}$$

$$\lim_{z\to z_0} j(z) = \sum_{n=0}^{\infty} a_n J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0), \tag{5.2}$$

provided $|z| < R$ *and* $z \in g_\varphi$.

The Proofs of of Theorem 4.1. and Theorem 5.1., using the specific properties of the considered functions, follow the lines of the analogous type theorems in [15]-[17].

6. (E, Z_0)-SUMMATIONS

Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, ... \tag{6.1}$$

Note that each of the functions $\widetilde{E}_{n,\beta}(z), \widetilde{E}_{\alpha,n}(z) \ \ (n \in \mathbb{N})$, $J^{\mu,\,m}_{n-2\lambda,\lambda}(z)$ being an entire function, not identically zero, has at most a finite number of zeros in the closed and bounded set $|z| \leq R$. Moreover, due to proven asymptotic formulae, only finite number of these functions may have some zeros at all except the zero.

Let $z_0 \in \mathbb{C}$, $|z_0| = R$, $0 < R < \infty$, $\widetilde{E}_{n,\beta}(z_0) \neq 0$, $\widetilde{E}_{\alpha,n}(z_0) \neq 0$ and $J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0) \neq 0$.

For the sake of brevity, denote

$$E^*_{n,\beta}(z; z_0) = \frac{\widetilde{E}_{n,\beta}(z)}{\widetilde{E}_{n,\beta}(z_0)}, \quad E^*_{\alpha,n}(z; z_0) = \frac{\widetilde{E}_{\alpha,n}(z)}{\widetilde{E}_{\alpha,n}(z_0)}. \tag{6.2}$$

$$J^*_{n,\lambda,\mu,m}(z; z_0) = \frac{J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}.$$

Further, we introduce the following new notion of summability, related to series in M-L functions.

Definition 6.1. The series (6.1) is said to be (J, z_0) - summable (respectively (E_β, z_0), (E_α, z_0) - summable) if the series

$$\sum_{n=0}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z; z_0) \tag{6.3}$$

respectively

$$\sum_{n=0}^{\infty} a_n E^*_{n,\beta}(z; z_0), \tag{6.4.β}$$

$$\sum_{n=0}^{\infty} a_n E^*_{\alpha,n}(z; z_0), \tag{6.4.α}$$

converge in the disk $|z| < R$ and, moreover, there exists the limit

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z; z_0), \tag{6.5}$$

respectively

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n E^*_{n,\beta}(z; z_0), \tag{6.6.β}$$

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n E^*_{\alpha,n}(z; z_0), \tag{6.6.α}$$

provided z remains on the segment $[0, z_0]$.

Remark 6.1. Every (J, z_0), $(E_\beta, z_0), (E_\alpha, z_0)$, - summation is regular, and this property is just a particular case of Theorem 5.1.

7. TAUBER TYPE THEOREMS

A Tauberian theorem is a statement that relates the Abel summability and the standard convergence of a number series by means of some assumptions imposed on the general term of the series under question. A classical result in this direction is given by Theorem 85 in Hardy [7].

In this paper we extend the validity of such type of assertion to series in Mittag-Leffler and Lommel-Wright functions, by means of the following theorem.

Theorem 7.1. (of Tauber type). *If the series* (6.1) *is* (J, z_0) *- summable, (resp.* (E_β, z_0) (E_α, z_0)*), and*

$$\lim_{n\to\infty} n a_n = 0, \tag{7.1}$$

then it is convergent.

Tauber type theorems have been given also for summations by means of Laguerre polynomials [20], and Bessel type functions by the author [15], [16].

8. LITTLEWOOD TYPE THEOREMS

At first sight, it seems that the condition $a_n = o(1/n)$ is essential. Nevertheless, Littlewood succeeds to weaken it and to obtain a stronger version of the Tauber theorem (see [7], Theorem 90).

A Littlewood generalization of the $o(n)$- version of the Tauber type theorem (Theorem 7.1) is given below. Similar theorems have been proved in [2] (a generalization of a Tauber type theorem, proven in [20]) and [17].

Theorem 8.1 (of Littlewood type). *If the series (6.1) is* (J, z_0) *- summable, (resp.* (E_β, z_0) (E_α, z_0)*), and*

$$a_n = O(1/n), \tag{8.1}$$

then the series (6.1) converges.

Proof. Let z belongs to the segment $[0, z_0]$. By the analogous way as in [17], the functions $E_{\alpha,n}(z)$ can be written in the following form

$$E_{\alpha,n}(z) = \frac{1}{\Gamma(n)} \ \ (1 + \theta_{\alpha,n}(z)), \ \ z \in \mathbb{C}.$$

Moreover there exists a constant C_1 so that for all the nonnegative integers n

$$|\theta_{\alpha,n}(z)| \leq C_1 \frac{\Gamma(n)}{\Gamma(\alpha+n)}, \quad \theta_{\alpha,n}(z) \to 0 \ \text{ as } n \to \infty, \tag{8.2}$$

on the compact subsets of $\mathbb{C}$. Obviously the functions $\theta_{\alpha,n}(z)$ are holomorphic for $z \in \mathbb{C}$ and the convergence is uniform on the compact subsets of the complex plane. Therefore we obtain

$$\begin{aligned} a_n E^*_{\alpha,n}(z; z_0) &= a_n \left(\frac{z}{z_0}\right)^n \frac{1+\theta_n(z)}{1+\theta_n(z_0)} \\ &= a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_n(z; z_0)\right), \end{aligned}$$

where $\widetilde{\theta}_n(z; z_0) = \dfrac{\theta_n(z) - \theta_n(z_0)}{1+\theta_n(z_0)}$. Then $\widetilde{\theta}_n(z; z_0) = O(1/n^\alpha)$, due to (8.2).

Let us write (6.4.α) in the form

$$\sum_{n=0}^{\infty} a_n E^*_{\alpha,n}(z;z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \left(1+\widetilde{\theta}_n(z;z_0)\right). \quad (8.3)$$

Denoting $w_n(z) = a_n \left(\frac{z}{z_0}\right)^n \widetilde{\theta}_n(z;z_0)$ we consider the series $\sum_{n=0}^{\infty} w_n(z)$. Since $|w_n(z)| \le |a_n|\,|\widetilde{\theta}_n(z;z_0)|$ and according to the conditions (8.1) and (8.2), there exists a constant C, such that $|w_n(z)| \le C/n^{1+\alpha}$. Since $\sum_{n=1}^{\infty} 1/n^{1+\alpha}$ converges, the series $\sum_{n=0}^{\infty} w_n(z)$ is also convergent, even absolutely and uniformly on the segment $[0, z_0]$. Therefore (since $\lim_{z\to z_0} w_n(z) = 0$)

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} w_n(z) = \sum_{n=0}^{\infty} \lim_{z\to z_0} w_n(z) = 0. \quad (8.4)$$

Obviously, the assumption that the series (6.1) is (E_α, z_0) - summable implies the existence of the limit (6.6.α). Then, having in mind that (8.3) can be written in the form

$$\sum_{n=0}^{\infty} a_n E^*_{\alpha,n}(z;z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n + \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \widetilde{\theta}_n(z;z_0),$$

we conclude that there exists the limit

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \quad (8.5)$$

and, moreover,

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n E^*_{\alpha,n}(z;z_0) = \lim_{z\to z_0} \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n. \quad (8.6)$$

From the existence of the limit (8.5) it follows that the series (6.1) is A-summable. Then according to Theorem 90 ([7]), the series (6.1) converges.

The other two cases (for (E_β, z_0) and (J, z_0) - summations) go analogously. ■

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